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B.Sc (part-I) Hons
paper-I

Forced oscillation and its Equation.

Forced oscillation:— When a body capable of oscillation is subjected to an external periodic force, it begins to oscillate under the action of applied force. In the beginning, the body tries to oscillate with natural frequency, while the external force tries to impose its own frequency upon the body. Thus there is a sort of tussel between the driver (external force) and the driven (body) during which the amplitude of oscillation rises and falls irregularly a number of times. This is 'transient effect' which soon dies away. Finally, the body yields to the external force and oscillates with constant amplitude and phase, and with the frequency of the force. Its oscillations are then called 'forced oscillations'. The oscillating body is called "driven harmonic oscillator" and the external force is called 'driving force'.

Equation of Forced oscillations:— Let us consider a system oscillating about an equilibrium position under an external periodic force. Let x be its displacement from the equilibrium position at an instant during the oscillation. Its instantaneous velocity dx/dt. The forces acting upon the system at this instant are:

(i) A restoring force proportional to the displacement but acting in the opposite direction. This may be written as

$-Kx$ (where K is the force Constant)

(ii) A frictional force proportional to the velocity but acting in the opposite direction. This may be written as

$-b\frac{dx}{dt}$ where b is positive Constant

(iii) An external periodic force represented by $F_0 \sin pt$

where F_0 is the maximum value of this force and p is its angular frequency. Thus the total force F acting upon the system is

$$F = -Kx - b\frac{dx}{dt} + F_0 \sin pt$$

By Newton's Second law this must be equal to the product of the mass m of the System and the instantaneous acceleration $\frac{d^2x}{dt^2}$

That is $-Kx - b\frac{dx}{dt} + F_0 \sin pt = m\frac{d^2x}{dt^2}$

$$\text{or, } \frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{K}{m}x = \frac{F_0}{m}\sin pt$$

putting $\frac{b}{m} = 2r$, $\frac{K}{m} = \omega^2$ and $\frac{F_0}{m} = f_0$

we get,

$$\frac{d^2x}{dt^2} + 2r\frac{dx}{dt} + \omega^2 x = f_0 \sin pt \quad \text{--- (1)}$$

This is the differential equation of motion of the forced harmonic oscillator.

Once the transient effect dies away and a Steady State is set up, the System will oscillate with the frequency of the applied force. This is a necessory physical Condition of the oscillation. Let us therefore try the following equation as a solution of equation ①

$$x = A\sin(pt-\theta) \quad \text{———} ②$$

Where A and θ are arbitrary constants.

Differentiating equation ② twice with respect to t, we get

$$\frac{dx}{dt} = pA\cos(pt-\theta)$$

and $$\frac{d^2x}{dt^2} = -p^2A\sin(pt-\theta)$$

Substituting the values of x, $\frac{dx}{dt}$ and $\frac{d^2x}{dt^2}$ in eqn ①

$$-p^2A\sin(pt-\theta) + 2\gamma pA\cos(pt-\theta) + \omega^2A\sin(pt-\theta) = f_0\sin\{(pt-\theta)+\theta\}$$

or, $$A(\omega^2-p^2)\sin(pt-\theta) + 2\gamma pA\cos(pt-\theta) = f_0\sin(pt-\theta)\cos\theta + f_0\cos(pt-\theta)\sin\theta$$

If this equation is to be satisfied for all value of t, then the coefficients of $\sin(pt-\theta)$ and $\cos(pt-\theta)$ on the two sides must be equal. Equating them we obtain

$$A(\omega^2-p^2) = f_0\cos\theta \quad \text{———} ③$$

and $$2\gamma pA = f_0\sin\theta \quad \text{———} ④$$

Squaring and adding equation ③ and ④ we get.

$$A^2\{(\omega^2-p^2)^2 + 4\gamma^2p^2\} = f_0^2$$

or, $$A = \frac{f_0}{\sqrt{[(\omega^2-p^2)^2+4\gamma^2p^2]}} \quad \text{———} ⑤$$

Dividing equation ④ by equation ③, we get

$$\tan\theta = \frac{2\gamma p}{\omega^2-p^2} \quad \text{———} ⑥$$

Substituting the value of A from equation ⑤ in equation ② we get

$$x = \frac{f_0}{\sqrt{[(\omega^2-p^2)^2+4\gamma^2p^2]}}\sin(pt-\theta) \quad \text{———} ⑦$$

Where θ is given by equation ⑥. This is the solution of the differential equation of the forced harmonic oscillator. This is shows that the oscillator has a constant amplitude A and a frequency equal to that of the impressed force, and lags behind the force in phase by θ.